On a representation of time space-harmonic polynomials via symbolic Lévy processes.

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Abstract

In this paper, we review the theory of time space-harmonic polynomials developed by using a symbolic device known in the literature as the classical umbral calculus. The advantage of this symbolic tool is twofold. First a moment representation is allowed for a wide class of polynomial stochastic involving the Lévy processes in respect to which they are martingales. This representation includes some well-known examples such as Hermite polynomials in connection with Brownian motion. As a consequence, characterizations of many other families of polynomials having the time space-harmonic property can be recovered via the symbolic moment representation. New relations with Kailath-Segall polynomials are stated. Secondly the generalization to the multivariable framework is straightforward. Connections with cumulants and Bell polynomials are highlighted both in the univariate case and in the multivariate one. Open problems are addressed at the end of the paper.

keywords: Lévy process, time-space harmonic polynomial, Kailath-Segall polynomial, cumulant, umbral calculus.

1 Introduction

In mathematical finance, a Lévy process [22] is usually employed to model option pricing.

Definition 1.1. *A Lévy process $X = \{X_t\}_{t \geq 0}$ is a stochastic process satisfying the following properties:*

a) *X has independent and stationary increments;*

b) $P[X(0) = 0] = 1$ *on the probability space* $(\Omega, \mathcal{F}, P)$;

c) *X is stochastically continuous, i.e. for all $a > 0$ and for all $s \geq 0$,* $\lim_{t \to s} P(|X(t) - X(s)| > a) = 0$.

The employment of Lévy processes in mathematical finance is essentially due to the property of manage continuous processes interspersed with jump discontinuities of random size and at random times, well fitting the main dynamics of a market. In order to include the risk neutrality, a martingale pricing could be applied to options. But Lévy processes do not necessarily share the martingale property unless they are centred. Instead of focusing the attention on the expectation, a different approach consists in resorting a family of stochastic processes, called *polynomial processes* and introduced very recently in [4]. These processes are built by considering a suitable family of polynomials $\{P(x,t)\}_{t\geq 0}$ and by replacing the indeterminate x with a stochastic process X_t. Introduced in [25] and called *time-space harmonic polynomials* (TSH), the polynomials $\{P(x,t)\}_{t\geq 0}$ are such that

$$E[P(X_t,t) \mid \mathfrak{F}_s] = P(X_s,s), \qquad \text{for } s \leq t \tag{1.1}$$

where $\mathfrak{F}_s = \sigma\left(X_\tau : \tau \leq s\right)$ is the natural filtration associated with $\{X_t\}_{t\geq 0}$.

As done in [19] for the discretized version of a Lévy process, that is a random walk, TSH polynomials can be characterized as coefficients of the Taylor expansion

$$\frac{\exp\{zX_t\}}{E[\exp\{zX_t\}]} = \sum_{k\geq 0} R_k(X_t,t)\frac{z^k}{k!} \tag{1.2}$$

in some neighborhood of the origin. The left-hand side of (1.2) is the so-called Wald's exponential martingale [17]. Wald's exponential martingale is well defined only when the process admits moment generating function $E[\exp\{zX_t\}]$ in a suitable neighborhood of the origin. Different authors have tried to overcome this gap by using other tools. Sengupta [25] uses a discretization procedure to extend the results proved by Goswami and Sengupta in [15]. Solé and Utzet [27] use Ito's formula showing that TSH polynomials with respect to Lévy processes are linked to the exponential complete Bell polynomials [3]. Wald's exponential martingale (1.2) has been recently reconsidered also in [26], but without this giving rise to a closed expression for these polynomials.

The employment of the classical umbral calculus turns out to be crucial in dealing with (1.2). Indeed, the expectation of the polynomial processes $R_k(X_t,t)$ can be considered without taking into account any question involving the convergence of the right hand side of (1.2). Indeed the family $\{R_k(x,t)\}_{t\geq 0}$ is linked to the Bell polynomials which are one of the building blocks of the symbolic method. The main point here is that any TSH polynomial could be expressed as a linear combination of the family $\{R_k(x,t)\}$ and the symbolic representation of these coefficients is particularly suited to be implemented in any symbolic software. The symbolic approach highlights the role played by Lévy processes with regard to which the property (1.1) holds and makes clear the dependence of this representation on their cumulants.

The paper is organized as follows. Section 2 is provided for readers unaware of the classical umbral calculus. We have chosen to recall terminology, notation

and the basic definitions strictly necessary to deal with the object of this paper. We skip any proof. The reader interested in is referred to [10, 11]. The theory of TSH polynomials is resumed in Section 3 together with the symbolic representation of Lévy processes closely related to their infinite divisible property. Umbral expressions of many classical families of polynomials as TSH polynomials with respect to suitable Lévy processes are outlined. The generalization to the multivariable framework is given in Section 4. This setting allows us to deal with multivariate Hermite, Euler and Bernoulli polynomials as well as with the class of multivariate Lévy-Sheffer systems introduced in [9]. Open problems are addressed at the end of the paper.

2 The classical umbral calculus.

Let $\mathbb{R}[x]$ be the ring of polynomials with real coefficients[1] in the indeterminate x. The classical umbral calculus is a syntax consisting in an alphabet $\mathcal{A} = \{\alpha, \beta, \gamma, \ldots\}$ of elements, called *umbrae*, and a linear functional $E : \mathbb{R}[x][\mathcal{A}] \longrightarrow \mathbb{R}[x]$, called *evaluation*, such that $E[1] = 1$ and

$$E[x^n \, \alpha^i \, \beta^j \, \cdots \, \gamma^k] = x^n \, E[\alpha^i] \, E[\beta^j] \, \cdots \, E[\gamma^k] \quad \text{(uncorrelation property)}$$

where $\alpha, \beta, \ldots, \gamma$ are distinct umbrae and $n, i, j, \ldots, k$ are nonnegative integers.

A sequence $\{a_i\}_{i \geq 0} \in \mathbb{R}[x]$, with $a_0 = 1$, is *umbrally represented*[2] by an umbra α if $E[\alpha^i] = a_i,$ for all nonnegative integers i. Then a_i is called the i-th *moment* of α. An umbra is *scalar* if its moments are elements of $\mathbb{R}$ while it is *polynomial* if its moments are polynomials of $\mathbb{R}[x]$. Special scalar umbrae are given in Table 1. The core of this moment symbolic calculus consists in

Umbrae	*Moments*
Augmentation ϵ	$E[\epsilon^i] = \delta_{0,i}$, with $\delta_{i,j} = 1$ if $i = j$, otherwise $\delta_{i,j} = 0$.
Unity u	$E[u^i] = 1$
Boolean unity $\bar{u}$	$E[\bar{u}^i] = i!$
Singleton χ	$E[\chi] = 1$ and $E[\chi^i] = 0$, for all $i > 1$
Bell β	$E[\beta^i] = B_i$, with B_i the i-th Bell number
Bernoulli ι	$E[\iota^i] = \mathfrak{B}_i$, with $\mathfrak{B}_i$ the i-th Bernoulli number
Euler ε	$E[\varepsilon^i] = \mathfrak{E}_i$, with $\mathfrak{E}_i$ the i-th Euler number

Table 1: Special scalar umbrae. The equalities on the right column refer to all nonnegative integer i, unless otherwise specified.

defining the *dot-product* of two umbrae, whose construction is shortly recalled in the following.

[1] The ring $\mathbb{R}[x]$ may be replaced by any ring in whatever number of indeterminates, as for example $\mathbb{R}[x, y, \ldots]$.

[2] When no misunderstanding occurs, we use the notation $\{a_i\}$ instead of $\{a_i\}_{i \geq 0}$

First let us underline that in the alphabet $\mathcal{A}$ two (or more) distinct umbrae may represent the same sequence of moments. More formally, two umbrae α and γ are said to be *similar* when $E[\alpha^n] = E[\gamma^n]$ for all nonnegative integers n, in symbols $\alpha \equiv \gamma$. Therefore, given a sequence $\{a_n\}$, there are infinitely many distinct, and thus similar umbrae, representing the sequence.

Denote $\alpha' + \alpha'' + \cdots + \alpha'''$ by the symbol $n.\alpha$, where $\{\alpha', \alpha'', \ldots, \alpha'''\}$ is a set of n uncorrelated umbrae similar to α. The symbol $n.\alpha$ is an example of *auxiliary umbra*. In a *saturated* umbral calculus, the auxiliary umbrae are managed as they were elements of $\mathcal{A}$ [21]. The umbra $n.\alpha$ is called the *dot-product* of the integer n and the umbra α with moments [11]:

$$q_i(n) = E[(n.\alpha)^i] = \sum_{k=1}^{i} (n)_k B_{i,k}(a_1, a_2, \ldots, a_{i-k+1}), \tag{2.1}$$

where $(n)_k$ is the lower factorial and $B_{i,k}$ are the exponential partial Bell polynomials [3].

In (2.1), the polynomial $q_i(n)$ is of degree i in n. If the integer n is replaced by $t \in \mathbb{R}$, in (2.1) we have $q_i(t) = \sum_{k=1}^{i} (t)_k B_{i,k}(a_1, a_2, \ldots, a_{i-k+1})$. Denote by $t.\alpha$ the auxiliary umbra such that $E[(t.\alpha)^i] = q_i(t)$, for all nonnegative integers i. The umbra $t.\alpha$ is the dot-product of t and α. A kind of distributive property holds:

$$(t+s).\alpha \equiv t.\alpha + s.\alpha', \quad s, t \in \mathbb{R} \tag{2.2}$$

where $\alpha' \equiv \alpha$. In particular if in (2.1) the integer n is replaced by $-t$, the auxiliary umbra $-t.\alpha$ is such that

$$-t.\alpha + t.\alpha' \equiv \epsilon, \tag{2.3}$$

where $\alpha' \equiv \alpha$. Due to equivalence (2.3), the umbra $-t.\alpha$ is the *inverse*[3] umbra of $t.\alpha$.

Let us consider again the polynomial $q_i(t)$ and suppose to replace t by an umbra γ. The polynomial $q_i(\gamma) \in \mathbb{R}[x][\mathcal{A}]$ is an *umbral polynomial* with support [4] $\mathrm{supp}\,(q_i(\gamma)) = \{\gamma\}$. The *dot-product of* γ *and* α is the auxiliary umbra $\gamma.\alpha$ such that $E[(\gamma.\alpha)^i] = E[q_i(\gamma)]$ for all nonnegative integers i. Two umbral polynomials p and q are said to be *umbrally equivalent* if $E[p] = E[q]$, in symbols $p \simeq q$. Therefore equations (2.1), with n replaced by an umbra γ, can be written as the equivalences

$$q_i(\gamma) \simeq (\gamma.\alpha)^i \simeq \sum_{k=1}^{i} (\gamma)_k B_{i,k}(a_1, a_2, \ldots, a_{i-k+1}). \tag{2.4}$$

[3]Since $-t.\alpha$ and $t.\alpha$ are two distinct symbols, they are considered uncorrelated, therefore $-t.\alpha + t.\alpha' \equiv -t.\alpha + t.\alpha \equiv \epsilon$. When no confusion occurs, we will use this last similarity instead of (2.3).

[4]The support $\mathrm{supp}\,(p)$ of an umbral polynomial $p \in \mathbb{R}[x][\mathcal{A}]$ is the set of all umbrae occurring in it.

Special dot-product umbrae are the α-cumulant umbra $\chi.\alpha$ and α-partition umbra $\beta.\alpha$, that we will use later on. In particular any umbra is a partition umbra [11]. This property means that if $\{a_i\}$ is a sequence umbrally represented by an umbra α, then there exists a sequence $\{h_i\}$ umbrally represented by an umbra κ_α, such that $\alpha \equiv \beta.\kappa_\alpha$. The umbra κ_α is similar to the α-cumulant umbra, that is $\kappa_\alpha \equiv \chi.\alpha$, and its moments share the well-known properties of cumulants. [5]

Dot-products can be nested. For example, moments of $(\alpha.\varsigma).\gamma$ can be recursively computed by applying two times formula (2.4). Parenthesis can be avoided since $(\alpha.\varsigma).\gamma \equiv \alpha.(\varsigma.\gamma)$. In particular $\alpha.\beta.\gamma$, with β the Bell umbra, is the so-called *composition umbra*, with moments

$$E[(\alpha.\beta.\gamma)^i] = \sum_{k=1}^{i} a_k B_{i,k}(g_1, g_2, \ldots, g_{i-k+1}), \tag{2.5}$$

where $\{a_i\}$ are moments of α and $\{g_i\}$ are moments of γ. When the umbra α is replaced by $t \in \mathbb{R}$, then equation (2.5) gives the i-th moment of a compound Poisson random variable (r.v.) of parameter t :

$$E[(t.\beta.\gamma)^i] = \sum_{k=1}^{i} t^k B_{i,k}(g_1, g_2, \ldots, g_{i-k+1}).$$

There are more auxiliary umbrae that will employed in the following. For example, if $E[\alpha] \neq 0$, the compositional inverse $\alpha^{<-1>}$ of an umbra α is such that $\alpha.\beta.\alpha^{<-1>} \equiv \alpha^{<-1>}.\beta.\alpha \equiv \chi$. The derivative of an umbra α is the umbra α_D whose moments are $E[\alpha_D^i] = i\, a_{i-1}$ for all nonnegative integers $i \geq 1$. The disjoint sum $\alpha \dot{+} \gamma$ of α and γ represents the sequence $\{a_i + g_i\}$. Its main property involves the Bell umbra:

$$\beta.(\alpha \dot{+} \gamma) \equiv \beta.\alpha + \beta.\gamma. \tag{2.6}$$

2.1 Symbolic Lévy processes.

The family of auxiliary umbrae $\{t.\alpha\}_{t \in I}$, with $I \subset \mathbb{R}^+$, is the umbral counterpart of a stochastic process $\{X_t\}_{t \in I}$ having all moments and such that $E[X_t^i] = E[(t.\alpha)^i]$ for all nonnegative integers i. This symbolic representation parallels the well-known infinite divisible property of a Lévy process, summarized by the following equality in distribution

$$X_t \stackrel{d}{=} \underbrace{\Delta X_{t/n} + \cdots + \Delta X_{t/n}}_{n} \tag{2.7}$$

[5]For cumulants $\{C_i(Y)\}$ of a random variable Y, the following properties hold for all nonnegative integers i : (Homogeneity) $C_i(aY) = a^i C_i(Y)$ for $a \in \mathbb{R}$, (Semi-invariance) $C_1(Y + a) = a + C_1(Y), C_i(Y + a) = C_i(Y)$ for $i \geq 2$, (Additivity) $C_i(Y_1 + Y_2) = C_i(Y_1) + C_i(Y_2)$, if Y_1 and Y_2 are independent random variables.

with $\Delta X_{t/n}$ a r.v. corresponding to the increment of the process over an interval of amplitude t/n. The n-fold convolution (2.7) is usually expressed by the product of n times a characteristic function $E[e^{\mathrm{i}zX_t}] = E[e^{\mathrm{i}z\Delta X_{t/n}}]^n$ with $\mathfrak{i}$ the imaginary unit. More generally one has

$$E[e^{\mathrm{i}zX_t}] = E[e^{\mathrm{i}zX_1}]^t. \tag{2.8}$$

Equation (2.8) allows us to show that the auxiliary umbra $t.\alpha$ is the symbolic version of X_t. To this aim we recall that the formal power series

$$f(\alpha, z) = 1 + \sum_{i \geq 1} a_i \frac{z^i}{i!} \tag{2.9}$$

is the generating function of an umbra α, umbrally representing the sequence $\{a_i\}$. Table 2 shows generating functions for some special auxiliary umbrae introduced in the previous section.

Umbrae	*Generating functions*
Augmentation ϵ	$f(\epsilon, z) = 1$
Unity u	$f(u, z) = e^z$
Boolean unity $\bar{u}$	$f(\bar{u}, z) = \frac{1}{1-z}$
Singleton χ	$f(\chi, z) = 1 + z$
Bell β	$f(\beta, z) = \exp[e^z - 1]$
Bernoulli ι	$f(\iota, z) = z/(e^z - 1)$
Euler η	$f(\eta, z) = 2\,e^z/[e^z + 1]$
dot-product $n.\alpha$	$f(n.\alpha, z) = f(\alpha, z)^n$
dot-product $t.\alpha$	$f(t.\alpha, z) = f(\alpha, z)^t$
dot-product $\gamma.\alpha$	$f(\gamma.\alpha, z) = f(\gamma, \log f(\alpha, z))$
α-cumulant $\chi.\alpha$	$f(\chi.\alpha, z) = 1 + \log[f(\alpha, z)]$
α-partition $\beta.\alpha$	$f(\beta.\alpha, z) = \exp[f(\alpha, z) - 1]$
composition $\alpha.\beta.\gamma$	$f(\alpha.\beta.\gamma, z) = f[\alpha, f(\gamma, z) - 1]$
α-partition $t.\beta.\gamma$	$f(t.\beta.\gamma, z) = \exp[t(f(\gamma, z) - 1)]$
derivative α_D	$f(\alpha_D, z) = 1 + z f(\alpha, z)$

Table 2: Generating functions for some special auxiliary umbrae.

As for infinitely divisible stochastic processes (2.7), one has $f(t.\alpha, z) = f(\alpha, z)^t$. It is well-known that the class of infinitely divisible distributions coincides with the class of limit distributions of compound Poisson distributions [14]. By the symbolic method, any Lévy process is of compound Poisson type [8]. This result is a direct consequence of the Lévy-Khintchine formula [22] involving the moment generating function of a Lévy process. Indeed, if $\phi(z, t)$ denotes the moment generating function of X_t and $\phi(z)$ denotes the moment generating function of X_1 then $\phi(z, t) = \phi(z)^t$ from (2.8). From the Lévy-Khintchine

formula $\phi(z) = \exp[g(z)]$, with

$$g(z) = zm + \frac{1}{2}s^2z^2 + \int_{\mathbb{R}} \left(e^{zx} - 1 - z\,x\,\mathbf{1}_{\{|x|\leq 1\}}\right) \mathrm{d}(\nu(x)). \tag{2.10}$$

The term (m, s^2, ν) is the Lévy triplet and ν is the Lévy measure. The function $\phi(z,t)$ shares the same exponential form of the moment generating function $f(t.\beta.\gamma, z)$ of a compound Poisson process, see Table 2. If ν admits all moments and if $c_0 = m + \int_{\{|x|\geq 1\}} x\,\mathrm{d}(\nu(x))$, then the function $g(z)$ given in (2.10) has the form

$$g(z) = c_0 z + \frac{1}{2}s^2z^2 + \int_{\mathbb{R}} (e^{zx} - 1 - z\,x)\,\mathrm{d}(\nu(x)). \tag{2.11}$$

Thanks to (2.11), the symbolic representation $t.\beta.\gamma$ of a Lévy process is such that the umbra γ can be further decomposed in a suitable disjoint sum of umbrae. Indeed, assume

i) ς an umbra with generating function $f(\varsigma, z) = 1 + z^2/2$,
ii) η an umbra with generating function $f(\eta, z) = \int_{\mathbb{R}} (e^{zx} - 1 - z\,x)\,\mathrm{d}(\nu(x))$.

Then a Lévy process is umbrally represented by the family

$$\{t.\beta.(c_0\chi \dot{+} s\varsigma \dot{+} \eta)\}_{t\geq 0} \quad \text{or} \quad \{t.\beta.(c_0\chi \dot{+} s\varsigma) + t.\beta.\eta\}_{t\geq 0}, \tag{2.12}$$

due to (2.6). Symbolic representation (2.12) is in agreement with Itô representation $X_t = W_t + M_t + c_0 t$ of a Lévy process with $W_t + c_0 t$ a Wiener process and M_t a compensated sum of jumps of a Poisson process involving the Lévy measure. Indeed the Gaussian component is represented by the symbol $t.\beta.(c_0\chi \dot{+} s\varsigma)$ as stated in [7], with c_0 corresponding to the mean and s^2 corresponding to the variance. The Poisson component is represented by $t.\beta.\eta$, that is $t.\beta.\eta$ is the umbral counterpart of a random sum $S_N = Y_1 + \cdots + Y_N$, with $\{Y_i\}$ independent and identically distributed r.v.'s corresponding to η, associated to the Lévy measure, and N a Poisson r.v. of parameter t. The representation $\{t.\beta.(c_0\chi \dot{+} s\varsigma \dot{+} \eta)\}_{t\geq 0}$ shows that the Lévy process is itself a compound Poisson process with $\{Y_i\}$ corresponding to the disjoint sum $(c_0\chi \dot{+} s\varsigma \dot{+} \eta)$.

More insights may be added on the role played by the umbra $c_0\chi \dot{+} s\varsigma \dot{+} \eta$. Indeed the moment generating function of a Lévy process can be written as $\phi(z,t) = \exp[t\,g(z)]$ with $g(z) = \log \phi(z)$. So the function $g(z)$ in (2.11) is the cumulant generating function of X_1 and $\gamma \equiv c_0\chi \dot{+} s\varsigma \dot{+} \eta$ is the symbolic representation of a r.v. whose moments are cumulants of X_1. Therefore, in the symbolic representation $t.\alpha$ of a Lévy process, introduced at the beginning of this section, the umbra α is the partition umbra of the cumulant umbra $\gamma \equiv c_0\chi \dot{+} s\varsigma \dot{+} \eta$ that is $\alpha \equiv \beta.\gamma$.

This remark suggests the way to construct the boolean and the free version of a Lévy process by using the boolean and the free cumulant umbra [12].

Boolean Lévy process. Let $M(z)$ be the ordinary generating function of a r.v. X, that is $M(z) = 1 + \sum_{i\geq 1} a_i z^i$, where $a_i = E[X^i]$.

The boolean cumulants of X are the coefficients b_i of the power series $B(z) = \sum_{i \geq 1} b_i z^i$ such that $M(z) = 1/[1 - B(z)]$. Denote by $\bar{\alpha}$ the umbra such that $E[\bar{\alpha}^i] = i! a_i$ for all nonnegative integers i. Then the umbra φ_{α} such that $\bar{\alpha} \equiv \bar{u}.\beta.\varphi_{\alpha}$ represents the sequence $\{b_i\}$ and is the α-boolean cumulant umbra. Therefore the symbolic representation of a boolean Lévy process is $t.\bar{u}.\beta.\varphi_{\alpha}$.

Free Lévy process. The noncrossing (or free) cumulants of X are the coefficients r_i of the ordinary power series $R(z) = 1 + \sum_{i \geq 1} r_i z^i$ such that $M(z) = R[zM(z)]$. If $\bar{\alpha}$ is the umbra with generating function $M(z)$, then the $\bar{\alpha}$-free cumulant umbra $\mathfrak{K}_{\bar{\alpha}}$ represents the sequence $\{i! r_i\}$. Assuming $\bar{\alpha}$ the umbral counterpart of the increment of a Lévy process over the interval $[0, 1]$, then the symbolic representation of a free Lévy process is $t.\bar{\mathfrak{K}}_{\alpha}.\beta.(-1.\bar{\mathfrak{K}}_{\alpha})_D^{<-1>}$.

Some more remarks on the parameters c_0 and s may be added. The Lévy process in (2.12) is a martingale if and only if $c_0 = 0$, see Theorem 5.2.1 in [1]. When this happens, $E[X_t] = 0$ for all $t \geq 0$ and the Lévy process is said to be centered. Since the parameter c_0 allows the contribution of the singleton umbra χ in (2.12), such an umbra plays a central role in the martingale property of a Lévy process. If $c_0 = 0$, no contribution is given by χ which indeed does not admit a probabilistic counterpart.

If $s = 0$, the corresponding Lévy process is a *subordinator*, with almost sure non-decreasing paths. The subordinator processes are usually employed to scale the time of a Lévy process. This device is useful to widen or to close the jumps of the paths in market dynamics. Denote by T_t the subordinator process of X_t chosen independent of X_t. The process X_{T_t} is of Lévy type too. The symbolic representation of T_t is $t.\beta.(c_0\chi \dot{+} \eta')$ so that $t.\beta.(c_0\chi \dot{+} \eta').\beta.(c_0\chi \dot{+} s\varsigma \dot{+} \eta)$ represents X_{T_t} with η and η' similar and uncorrelated umbrae. Despite its nested representation, the following result is immediately recovered: the process X_{T_t} is a compound Poisson process S_N with Y_i a randomized compound Poisson r.v. of random parameter η', shifted of c_0 in its mean.

3 Time-space harmonic polynomials.

Set $\mathcal{X} = \{\alpha\}$. The conditional evaluation $E(\cdot \,|\, \alpha)$ with respect to α handles the umbra α as it was an indeterminate [8]. In particular, $E(\cdot \,|\, \alpha) : \mathbb{R}[x][\mathcal{A}] \to \mathbb{R}[\mathcal{X}]$ is such that $E(1 \,|\, \alpha) = 1$ and

$$E(x^m \alpha^n \gamma^i \xi^j \cdots \,|\, \alpha) = x^m \alpha^n E[\gamma^i] E[\xi^j] \cdots$$

for uncorrelated umbrae $\alpha, \gamma, \xi, \ldots$ and for nonnegative integers $m, n, i, j, \ldots$. As it happens in probability theory, the conditional evaluation is an element of $\mathbb{R}[x][\mathcal{A}]$ and, if we take the overall evaluation of $E(p \,|\, \alpha)$, this gives $E[p]$, with $p \in \mathbb{R}[x][\mathcal{A}]$, that is $E[E(p \,|\, \alpha)] = E[p]$. Umbral polynomials p, not having α in its support, are such that $E(p \,|\, \alpha) = E[p]$.

Conditional evaluations with respect to auxiliary umbrae need to be handled carefully. For example, since $(n+1).\alpha \equiv n.\alpha + \alpha'$, the conditional evaluation with respect to the dot product $n.\alpha$ is defined as

$$E[(n+1).\alpha \,|\, n.\alpha] = n.\alpha + E[\alpha'],$$

and more general, from (2.2) with t and s replaced by n and m,
(3.1)

$$E([(n+m).\alpha]^k \mid n.\alpha) = E([n.\alpha + m.\alpha']^k \mid n.\alpha) = \sum_{j=0}^{k} \binom{k}{j} (n.\alpha)^j E[(m.\alpha')^{k-j}],$$

for all nonnegative integers n and m. Therefore, for $t \geq 0$ the conditional evaluation of $t.\alpha$ with respect to the auxiliary umbra $s.\alpha$, with $0 \leq s \leq t$, is defined according to (3.1) such as

$$E[(t.\alpha)^k \mid s.\alpha] = \sum_{j=0}^{k} \binom{k}{j} (s.\alpha)^j E([(t-s).\alpha']^{k-j}).$$

Equation (1.1) traces the way to extend the definition of polynomial processes to umbral polynomials.

Definition 3.1. *Let $\{P(x,t)\} \in \mathbb{R}[x][\mathcal{A}]$ be a family of polynomials indexed by $t \geq 0$. $P(x,t)$ is said to be a* TSH polynomial *with respect to the family of umbral polynomials $\{q(t)\}_{t\geq 0}$ if and only if $E\left[P(q(t),t) \,|\, q(s)\right] = P(q(s),s)$ for all $0 \leq s \leq t$.*

The main result of this section is the following theorem [8].

Theorem 3.2. *For all nonnegative integers k, the family of polynomials*[6]

$$Q_k(x,t) = E[(x - t.\alpha)^k] \in \mathbb{R}[x]$$

is TSH with respect to $\{t.\alpha\}_{t\geq 0}$.

By expanding $Q_k(x,t)$ via the binomial theorem, one has

$$Q_k(x,t) = \sum_{j=0}^{k} \binom{k}{j} x^j E[(-t.\alpha)^{k-j}]$$

so that

$$Q_k(t.\alpha,t) = \sum_{j=0}^{k} \binom{k}{j} (t.\alpha)^j E[(-t.\alpha)^{k-j}].$$

Since $t.\alpha$ is the symbolic version of a Lévy process, the property

$$E[Q_k(t.\alpha,t) \mid s.\alpha] = \sum_{j=0}^{k} \binom{k}{j} E[(t.\alpha)^j | s.\alpha] E[(-t.\alpha)^{k-j}] = Q_k(s.\alpha, s)$$

[6]When no confusion occurs, we will use the notation $x - t.\alpha$ to denote the polynomial umbra $-t.\alpha + x = x + (-t).\alpha$.

parallels equation (1.1). In particular $\{Q_k(x,t)\}$ is a polynomial sequence umbrally represented by the polynomial umbra $x - t.\alpha$, which is indeed the TSH polynomial umbra with respect to $t.\alpha$. Polynomial umbrae of type $x+\alpha$ are Appell umbrae [6]. Then $\{Q_k(x,t)\}$ is an Appell sequence and

$$\frac{\mathrm{d}}{\mathrm{d}x} Q_k(x,t) = k\, Q_{k-1}(x,t), \qquad \text{for all integers } k \geq 1.$$

The generating function of the TSH polynomial umbra $x - t.\alpha$ is

$$f(x - t.\alpha, z) = \frac{\exp\{xz\}}{f(\alpha,z)^t} = \sum_{k\geq 0} Q_k(x,t) \frac{z^k}{k!}. \tag{3.2}$$

By replacing x with $t.\alpha$ in (3.2), Wald's exponential martingale (1.2) is recovered. Equality of two formal power series is given in terms of equality of their corresponding coefficients, so that $E[R_k(X_t,t)] = E[Q_k(t.\alpha,t)]$ by comparing (3.2) with (1.2). Wald's identity $\sum_{k\geq 0} E[R_k(X_t,t)] z^k/k! = 1$ is encoded by the equivalence $t.\alpha - t.\alpha \equiv \epsilon$ obtained from $x - t.\alpha$ when x is replaced by $t.\alpha$.

The next proposition gives the way to compute the coefficients of $Q_k(x,t)$ in any symbolic software.

Proposition 3.3. *If $\{a_n\}$ is the sequence umbrally represented by the umbra α and $\{Q_k(x,t)\}$ is the sequence of TSH polynomials with respect to $\{t.\alpha\}_{t\geq 0}$, then*

$$Q_k(x,t) = \sum_{j,i=0}^{k} c_{i,j}^{(k)}\, t^i\, x^j,$$

with

$$c_{i,j}^{(k)} = \binom{k}{j} \sum_{\lambda \vdash k-j} \mathrm{d}_\lambda (-1)^{2l(\lambda)+i}\, s[l(\lambda), i]\, a_1^{r_1} a_2^{r_2} \cdots$$

where the sum is over all partitions[7] $\lambda = (1^{r_1}, 2^{r_2}, \ldots) \vdash k-j$, $s[l(\lambda),i]$ *denotes a Stirling number of first kind and* $d_\lambda = i!/(r_1! r_2! \cdots (1!)^{r_1} (2!)^{r_2} \cdots)$.

More properties on the coefficients of $Q_k(x,t)$ are given in [8].

Any TSH polynomial is a linear combination of $\{Q_k(x,t)\}$, which indeed are a bases of the space of TSH polynomials. The following theorem characterizes the coefficients of any TSH polynomial $P(x,t)$ in terms of the coefficients of $\{Q_k(x,t)\}$.

---

[7] Recall that a partition of an integer i is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$, where λ_j are weakly decreasing positive integers such that $\sum_{j=1}^m \lambda_j = i$. The integers λ_j are named *parts* of λ. The *length* of λ is the number of its parts and will be indicated by $l(\lambda)$. A different notation is $\lambda = (1^{r_1}, 2^{r_2}, \ldots)$, where r_j is the number of parts of λ equal to j and $r_1 + r_2 + \cdots = l(\lambda)$. Note that r_j is said to be the multiplicity of j. We use the classical notation $\lambda \vdash i$ to denote "λ is a partition of i".

Theorem 3.4. *A polynomial $P(x,t) = \sum_{j=0}^{k} p_j(t)\, x^j$ of degree k for all $t \geq 0$ is a TSH polynomial with respect to $\{t.\alpha\}_{t \geq 0}$ if and only if*

$$p_j(t) = \sum_{i=j}^{k} \binom{i}{j} p_i(0)\, E[(-t.\alpha)^{i-j}], \quad \textit{for } j = 0, \ldots, k.$$

3.1 Cumulants.

A different symbolic representation of TSH polynomials $\{Q_k(x,t)\}$ is

$$Q_k(x,t) = E[(x - t.\beta.\kappa_\alpha)^k],$$

with κ_α the α-cumulant umbra. The umbra $-t.\beta.\kappa_\alpha \equiv t.\beta.(-1.\kappa_\alpha) \equiv t.\beta.\kappa_{(-1.\alpha)}$ is the symbolic version of a Lévy process with sequence of cumulants of X_1 umbrally represented by $\kappa_{(-1.\alpha)}$. Therefore, also the polynomials $Q_k(x,t) = E[(x + t.\beta.\kappa_\alpha)^k]$ are TSH with respect to Lévy processes umbrally represented by $\{t.\beta.\kappa_{(-1.\alpha)}\}_{t\geq 0} \equiv \{t.(-1.\alpha)\}_{t \geq 0}$.

Moments of polynomial umbrae $t.\beta.\gamma$ involve the exponential Bell polynomials. When t is set equal to 1, then complete exponential Bell polynomials are recovered. More generally, the moments of $x - t.\beta.\kappa_\alpha \equiv x - t.\alpha$ can be expressed by using exponential complete Bell polynomials too since

(3.3)
$$x - t.\alpha \equiv \beta.[\chi.(x - t.\alpha)] \equiv \beta.\kappa_{x-t.\alpha} \equiv \beta.(\kappa_{(x.u)} \dot{+} \kappa_{(-t.\alpha)}) \equiv \beta.\kappa_{(x.u)} + \beta.\kappa_{(-t.\alpha)}$$

where $\kappa_{x-t.\alpha}$ is the cumulant umbra of $x - t.\alpha$, that could be replaced by $\kappa_{(x.u)} \dot{+} \kappa_{(-t.\alpha)}$ due to the additivity property of cumulants. The last equivalence in (3.3) follows from equivalence (2.6). From equivalences (3.3), we have

$$Q_k(x,t) = Y_k(x + h_1, h_2, \ldots, h_k), \tag{3.4}$$

with Y_k exponential complete Bell polynomials and $\{h_i\}$ cumulants of $-t.\alpha$. Equation (3.4) has been proved in [27] by using Teugel martingales.

For $Q_k(x,t)$, the Sheffer identity with respect to t holds:

$$Q_k(x, t+s) = \sum_{j=0}^{k} \binom{k}{j} P_j(s) Q_{k-j}(x,t),$$

where $P_j(s) = Q_j(0,s)$ for all nonnegative integers j.

3.2 Examples.

The discretized version of a Lévy process is a random walk $S_n = X_1 + X_2 + \cdots + X_n$, with $\{X_i\}$ independent and identically distributed r.v.'s. For the symbolic representation of a Lévy process we have dealt with, the symbolic counterpart of a random walk is the auxiliary umbra $n.\alpha$. Indeed the infinite divisible property (2.7) is highlighted in the summation $\alpha' + \alpha'' + \cdots + \alpha'''$,

encoded in the symbol $n.\alpha$, with $\alpha', \alpha'', \ldots, \alpha'''$ uncorrelated and similar umbrae. Nevertheless not all r.v.'s having the symbolic representation $n.\alpha$ share the infinite divisible property. For example, the binomial r.v. has not the infinite divisible property [22], nevertheless its symbolic representation is of type $n.\alpha$ where $\alpha \equiv \chi.p.\beta$ and $p \in (0,1)$. So the generality of the symbolic approach lies in the circumstance that if the parameter n is replaced by t, that is if the random walk is replaced by a Lévy process, more general classes of polynomials can be recovered for which many of the properties here introduced still hold. The following tables resume the TSH representation for different families of classical polynomials, see [20]. In particular Table 3 gives the umbra corresponding to the r.v. X_i of S_n in the first column, its umbral counterpart in the second column and the associated TSH polynomial in the third column. In Table 4, the TSH polynomials given in Table 3 are traced back to special families of polynomials. In particular, with the polynomials $\mathcal{P}_k(x,t)$ we refer to

$$\mathcal{P}_k(x,t) = \sum_{j=1}^{k} Q_j(x,t) B_{k,j}(m_1, m_2, \ldots, m_{k-j+1})$$

for suitable $\{Q_j(x,t)\}$ and $\{m_i\}$.

X_i	Umbral counterpart	Corresponding TSH polynomial
Uniform $[0,1]$	$-1.\iota$	$E[(x + n.\iota)^k]$
Bernoulli $p = \frac{1}{2}$	$\frac{1}{2}(-1.\epsilon + u)$	$E[(x + n.\left[\frac{1}{2}(-1.u + \epsilon)\right])^k]$
Bernoulli $p \in (0,1)$	$\chi.p.\beta$	$E[(x - n.\chi.p.\beta)^k]$
Sum of $a \in \mathbb{N}$ uniform r.v.'s on $[0,1]$	$a.(-1.\iota)$	$E[(x + (a\,n).\iota)^k]$

Table 3: TSH polynomials associated to special random walks

X_i	Special families of polynomials	Connection with TSH polynomials
Uniform $[0,1]$	Bernoulli $B_k(x,n)$	$B_k(x,t) = Q_k(x,t)$
Bernoulli $p = 1/2$	Euler $\mathcal{E}_k(x,n)$	$\mathcal{E}_k(x,n) = Q_k(x,t)$
Bernoulli $p \in (0,1)$	Krawtchouk $\mathcal{K}_k(x,p,n)$	$(n)_k \mathcal{K}_k(x,p,n) = \mathcal{P}_k(x,t)$ $m_i = E[((-1.\chi.p.\beta)^{<-1>})^i]$
Sum of $a \in \mathbb{N}$ uniform r.v.'s on $[0,1]$	pseudo-Narumi $N_k(x,an)$	$k!\,N_k(x,an) = \mathcal{P}_k(x,t)$ $m_i = E[(u^{<-1>})^i]$

Table 4: Connection between special families of polynomials and TSH polynomials

Next tables 5 and 6 give TSH polynomials for some special Lévy processes.

Lévy process	Umbral representation	TSH polynomial $Q_k(x,t)$
Brownian motion with variance s^2	$t.\beta.(s\varsigma)$	$E[(x - t.\beta.(s\varsigma))^k]$
Poisson process with parameter λ	$(t\lambda).\beta$	$E[(x - (t\lambda).\beta)^k]$
Gamma process with scale parameter 1 and shape parameter 1	$t.\bar{u}$	$E[(x - t.\bar{u})^k]$
Gamma process with scale parameter λ and shape parameter 1	$(t\lambda).\bar{u}$	$E[(x - (t\lambda).\bar{u})^k]$
Pascal process with parameter $d = p/q$ and $p + q = 1$	$t.\bar{u}.d.\beta$	$E[(x - t.\bar{u}.d.\beta)^k]$

Table 5: TSH polynomials associated to special Lévy processes

3.3 Orthogonality of TSH polynomials.

A special class of TSH polynomials is the one including the Lévy-Sheffer polynomials, whose applications within orthogonal polynomials are given in [23]. A sequence of polynomials $\{V_k(x,t)\}_{t\geq 0}$ [24] is a Lévy-Sheffer system if its generating function is such that

$$\sum_{k\geq 0} V_k(x,t)\frac{z^k}{k!} = (g(z))^t \exp\{xu(z)\}, \tag{3.5}$$

where $g(z)$ and $u(z)$ are analytic functions in a neighborhood of $z = 0$, $u(0) = 0$, $g(0) = 1$, $u'(0) \neq 0$ and $1/g(\tau(z))$ is an infinitely divisible moment generating function, with $\tau(z)$ such that $\tau(u(z)) = z$. Assume α an umbra such that $f(\alpha, z) = g(z)$ and γ an umbra such that $f(\gamma, z) = 1 + u(z)$. From (3.5), the Lévy-Sheffer polynomials are moments of $x.\beta.\gamma + t.\alpha$:

$$V_k(x,t) = E[(x.\beta.\gamma + t.\alpha)^k]. \tag{3.6}$$

Theorem 3.5. *The TSH polynomials $Q_k(x,t)$ are special Lévy-Sheffer polynomials.*

The proof of Theorem 3.5 is straightforward by choosing in (3.6) as umbra α its inverse $-1.\alpha$ and as umbra γ the singleton umbra χ. All the Lévy-Sheffer polynomials possess the TSH property. Indeed the following theorem has been proved in [8].

Theorem 3.6. *The Lévy-Sheffer polynomials $\{V_k(x,t)\}_{t\geq 0}$ are TSH with respect to Lévy processes umbrally represented by $\{-t.\alpha.\beta.\gamma^{<-1>}\}_{t\geq 0}$.*

Lévy process	Special family of polynomials	Connection with TSH polynomials
Brownian motion with variance s^2	Hermite $H_k^{(s^2)}(x)$	$H_k^{(s^2)}(x) = Q_k(x,t)$
Poisson process with parameter λ	Poisson-Charlier $\tilde{C}_k(x,\lambda t)$	$\tilde{C}_k(x,\lambda t) =$ $= \sum_{j=1}^{k} s(k,j) Q_k(x,t)$ with $s(k,j)$ Stirling numbers of first kind
Gamma process with scale parameter 1 and shape parameter 1	Laguerre $\mathcal{L}_k^{t-k}(x)$	$k!(-1)^k \mathcal{L}_k^{t-k}(x) = Q_k(x,t)$
Gamma process with scale parameter λ and shape parameter 1	actuarial $g_k(x,t)$	$g_k(x,t) = \mathcal{P}_k(x,t)$ $m_i = E[((\chi.(-\chi))^{<-1>})^i]$
Pascal process with parameter $d = p/q$ and $p+q=1$	Meixner polynomials of first kind $M_k(x,t,p)$	$(-1)^k (t)_k M_k(x,t,p) =$ $= \mathcal{P}_k(x,t)$ $m_i = E[(\chi.(-1.\chi + \chi/p))^i]$

Table 6: Special families of polynomials and TSH polynomials

In particular, one has [8]

$$(3.7) \qquad V_k(x,t) = \sum_{i=0}^{k} E[(x + t.\beta.\kappa_{(\alpha.\beta.\gamma^{<-1>})})^i] B_{k,i}(g_1, \ldots, g_{k-i+1}),$$

where $g_i = E[\gamma^i]$, for all nonnegative i, and $\kappa_{(\alpha.\beta.\gamma^{<-1>})}$ is the cumulant umbra of $\alpha.\beta.\gamma^{<-1>}$, with $\gamma^{<-1>}$ the compositional inverse of the umbra γ. When the umbra α is replaced by its inverse and the umbra γ by the singleton umbra, since $\chi^{<-1>} \equiv \chi$, the only contribution in the summation (3.7) is given by $i = k$. So again equation (3.7) reduces to $Q_k(x,t) = E[(x - t.\alpha)^k]$ since $\kappa_{(\alpha.\beta.\gamma^{<-1>})} \equiv \chi. - 1.\alpha$.

Within Lévy-Sheffer polynomials, the Lévy-Meixner polynomials are those orthogonal with respect to the Lévy processes $-t.\alpha.\beta.\gamma^{<-1>}$, due to their TSH property. The orthogonal property is

$$E\left[V_n(-t.\alpha.\beta.\gamma^{<-1>}, t) V_m(-t.\alpha.\beta.\gamma^{<-1>}, t)\right] = c_m \delta_{n,m}.$$

According to [23], all the polynomials in Table 6 are orthogonal. Their measure of orthogonality corresponds to the Lévy process $-t.\alpha$ since $\chi^{<-1>} \equiv \chi$ and $-t.\alpha.\beta.\gamma^{<-1>} \equiv -t.\alpha$.

3.4 Kailath-Segall polynomials.

Equivalence (3.3) gives the connection between TSH polynomials and Kailath-Segall polynomials [16], which is a different class of polynomials strictly related to Lévy processes. Indeed, both have a representation in terms of partition umbra of a suitable polynomial umbra. Overlaps are removed by suitably choosing the indeterminates.

The n-th Kailath-Segall polynomial $P_n(x_1, \ldots, x_n)$ is a multivariable polynomial such that when the indeterminates are replaced by the sequence $X_t^{(1)}, \ldots, X_t^{(n)}$ of variations of a Lévy process X_t

$$X_t^{(1)} = X_t, \;\; X_t^{(2)} = [X, X]_t, \;\; X_t^{(n)} = \sum_{s \geq t} (\Delta X_s)^n \;\; n \geq 3,$$

its iterated integrals are recovered

$$P_t^{(0)} = 1, \;\; P_t^{(1)} = X_t, \;\; P_t^{(n)} = \int_0^t P_{s-}^{(n-1)} \mathrm{d} X_s, \;\; n \geq 2,$$

that is $P_t^{(n)} = P_n\left(X_t^{(1)}, \ldots, X_t^{(n)}\right)$. The following recursion formula is known as Kailath-Segall formula

$$P_t^{(n)} = \frac{1}{n}\left(P_t^{(n-1)} X_t^{(1)} - P_t^{(n-2)} X_t^{(2)} + \cdots + (-1)^{n+1} P_t^{(0)} X_t^{(n)}\right). \tag{3.8}$$

When $X_t^{(1)}, \ldots, X_t^{(n)}$ are replaced by the power sums $S_1, \ldots, S_n$ in the indeterminates $x_1, \ldots, x_k$, according to formula (1.2) in [28] and Theorem 3.1 in [13], the corresponding polynomials $P_n(S_1, \ldots, S_n)$ are such that

$$n! P_n(S_1, \ldots, S_n) = E[(\beta.[(\chi.\chi)\sigma])^n], \tag{3.9}$$

where σ is the power sum umbra representing $\{S_j\}$ and the χ-cumulant umbra $\chi.\chi$ represents the sequence $\{(-1)^{i-1}(i-1)!\}$.

In order to recognize special TSH polynomials within the family $\{P_n\}$, two steps are necessary:

> *i)* Kailath-Segall polynomials need to be umbrally represented when the power sums $\{S_j\}$ are replaced by the indeterminates $\{x_i\}$;
> *ii)* the indeterminates $\{x_i\}$ need to be replaced by suitable terms involving x and t.

For the first step, we will use equation (3.9). Assume p an umbra representing the sequence $\{E[(\chi.x_i.\beta)^i]\}$. Observe that $E[(\chi.x_i.\beta)^i] = x_i$ for all nonnegative i. Then from (3.9) one has $n! P_n(x_1, \ldots, x_n) = E[(\beta.[(\chi.\chi)p\,])^n]$ so that the generating function of P_n is

$$f\left(\beta.[(\chi.\chi)p\,], z\right) = \exp\left(\sum_{n \geq 1} \frac{(-1)^{n+1}}{n} z^n x_n\right),$$

see also [30]. The strength of this symbolic representation essentially relies on the properties of the partition umbra $\beta.[(\chi.\chi)p]$ reproducing those of Bell polynomials. For example, the following property of Kailath-Segall polynomials

$$P_n(ax_1, a^2x_2, \ldots, a^nx_n) = a^n P_n(x_1, x_2, \ldots, x_n), \quad a \in \mathbb{R} \tag{3.10}$$

is proved by observing that $\beta.[a(\chi.\chi)p] \equiv a(\beta.[(\chi.\chi)p])$. For the next step, we need to characterize the indeterminates $x_1, x_2, \ldots$ such that

$$\kappa_{(x.u)} \dot{+} \kappa_{(-t.\alpha)} \equiv (\chi.\chi)p \Rightarrow E[(\kappa_{(x.u)})^n] + E[(\kappa_{(-t.\alpha)})^n] = (-1)^{n-1}(n-1)!\, x_n. \tag{3.11}$$

In the following we show some examples of how to perform this selection. These results extend the connections between TSH polynomials and Kailath-Segall polynomials analyzed in [27].

Generalized Hermite polynomials:

Since $E[(\kappa_{(x.u)})^i] = x\,\delta_{i,1}$ and $E[(\kappa_{(-t.\beta.(s\varsigma))})^i] = s^2\,t\,\delta_{i,2}$, then

$$k!P_k(x, s^2t, 0, \ldots, 0) = H_k^{(t)}(x)$$

where $\sum_{k\geq 0} H_k^{(t)}(x) z^k/k! = \exp\{xz - tz^2/2\}$.

Poisson-Charlier polynomials:

Poisson-Charlier polynomials $\{\widetilde{C}_k(x,t)\}$, with generating function

$$\sum_{k\geq 0} \widetilde{C}_k(x,t)\frac{z^k}{k!} = e^{-tz}(1+z)^x,$$

are umbrally represented by

$$\widetilde{C}_k(x, \lambda t) = E[(x.\chi - t.\lambda.u)^k], \tag{3.12}$$

see [8]. Nevertheless (3.12) differs from the result of Theorem 3.2, the TSH property holds since $\{\widetilde{C}_k(x,\lambda t)\}$ are a linear combination of special $Q_k(x,t)$. Moreover representation (3.12) allows us the connection with Kailath-Segall polynomials, when the indeterminates $\{x_i\}$ are chosen such that $E[(\kappa_{(x.\chi)})^i] + E[(\kappa_{(-t.\lambda.u)})^i] = (-1)^{i-1}(i-1)!\,x_i$. Since

$$E[(\kappa_{(x.\chi)})^i] + E[(\kappa_{(-t.\lambda.u)})^i] = \begin{cases} x - t\lambda, & i = 1, \\ (-1)^{i-1}(i-1)!\,x^i, & i \geq 2, \end{cases}$$

then $k!P_k(x - t\lambda, x, x, \ldots) = \widetilde{C}_k(x, \lambda t)$.

Laguerre polynomials:

Laguerre polynomials $\{\mathcal{L}_k^{t-k}(x)\}$ are TSH polynomials such that

$$k!(-1)^k \mathcal{L}_k^{t-k}(x) = E[(x - t.\bar{u})^k].$$

They can be traced back to Kailath-Segall polynomials if the indeterminates $\{x_i\}$ are characterized by $E[(\kappa_{(x.u)})^i] + E[(\kappa_{(-t.\bar{u})})^i] = (-1)^{i-1}(i-1)!\,x_i$. Since $f(\kappa_{(-t.\bar{u})}, z) = 1 + t\,\log(1-z)$ then $E[(\kappa_{(-t.\bar{u})})^i] = -t(i-1)!$. So

$$P_k(x - t, t, -t, t, \ldots) = (-1)^k \mathcal{L}_k^{t-k}(x)$$

and from (3.10) we have $P_k(t-x,t,t,\ldots) = \mathcal{L}_k^{t-k}(x)$.

Actuarial polynomials:

The actuarial polynomials $g_k(x,t)$ are a linear combination of suitable TSH polynomials $Q_k(x,t)$ (see Table 6) but they are moments of the polynomial umbra $\lambda t - x.\beta$, that is $g_k(x,t) = E[(\lambda t - x.\beta)^k]$. In order to characterize the connection with Kailath-Segall polynomials, the indeterminates $\{x_i\}$ need to be characterized by $E[(\kappa_{(\lambda t.u)})^i] + E[(\kappa_{(-x.\bar{u})})^i] = (-1)^{i-1}(i-1)!\, x_i$. As before $E[(\kappa_{(\lambda t.u)})^i] = \lambda\, t\, \delta_{i,1}$, instead $E[(\kappa_{(-x.\bar{u})})^i] = -x(i-1)!$ for all nonnegative integers i as in the previous example. Therefore one has $k!(-1)^k P_k(x-\lambda t, x, x, \ldots) = g_k(x,t)$.

Meixner polynomials of first kind:

Meixner polynomials of first kind $\{M_k(x,t,p)\}$ [23] are a linear combination of suitable TSH polynomials $Q_k(x,t)$ (see Table 6) but they are moments of the following polynomial umbra

$$(-1)^k (t)_k M_k(x,t,p) = E\left\{\left[x.\left(-1.\chi + \frac{\chi}{p}\right) - t.\chi\right]^k\right\},$$

which allows us to find the connection with Kailath-Segall polynomials. Indeed for all nonnegative integers i we have

$$E\left[\left\{\kappa_{x.\left(-1.\chi+\frac{\chi}{p}\right)}\right\}^i\right] = (-1)^{i-1}\,(i-1)!\,x\,\left(\frac{1}{p^i}-1\right)$$

and

$$E\left[\left\{\kappa_{(\chi.-t.\chi)}\right\}^i\right] = (-1)^{i-1}(i-1)!\,t.$$

Then Kailath-Segall polynomials give the Meixner polynomials $(-1)^k(t)_k M_k(x,t,p)$ by choosing

$$x_i = \left[\left(\frac{1}{p^i}-1\right)x - t\right]$$

for $i = 1, 2, \ldots$.

4 Symbolic multivariate Lévy processes.

In the multivariate case, the main device of the symbolic method here proposed relies on the employment of multi-indices of length d. Sequences like $\{g_{i_1,i_2,\ldots,i_d}\}$ are replaced with a product of powers $\mu_1^{i_1}\mu_2^{i_2}\cdots\mu_d^{i_d}$, where $(\mu_1,\mu_2,\ldots,\mu_d)$ are umbral monomials and $(i_1,i_2,\ldots,i_d)$ are nonnegative integers. Since the umbral monomials could not have disjoint support, then the evaluation E does not necessarily factorizes on the product $\mu_1^{i_1}\mu_2^{i_2}\cdots\mu_d^{i_d}$, that is

$$E[\mu_1^{i_1}\mu_2^{i_2}\cdots\mu_d^{i_d}] = E[\boldsymbol{\mu}^{\boldsymbol{i}}] = g_{\boldsymbol{i}} \tag{4.1}$$

where $\boldsymbol{i} = (i_1,i_2,\ldots,i_d)$ and $\boldsymbol{\mu} = (\mu_1,\mu_2,\ldots,\mu_d)$. We assume $g_{\mathbf{0}} = 1$ with $\mathbf{0} = (0,0,\ldots,0)$. Then $g_{\boldsymbol{i}}$ is called the multivariate moment of $\boldsymbol{\mu}$. Table 7 shows some special d-tuples we will use later.

	d-tuple	Generating functions
Multivariate Unity $\boldsymbol{u}$	$(u, \ldots, u')$	$f(\boldsymbol{u}, \boldsymbol{z}) = e^{z_1+\cdots+z_d}$.
Multivariate Gaussian $\boldsymbol{\varsigma}$	$(\varsigma, \ldots, \varsigma')$	$f(\boldsymbol{\varsigma}, \boldsymbol{z}) = 1 + \frac{1}{2}\boldsymbol{z}\boldsymbol{z}^T$.
Multivariate Bernoulli $\boldsymbol{\iota}$	$(\iota, \ldots, \iota)$	$f(\boldsymbol{\iota}, \boldsymbol{z}) = \dfrac{z_1 + \cdots + z_d}{e^{z_1+\cdots+z_d} - 1}$.
Multivariate Euler $\boldsymbol{\eta}$	$(\eta, \ldots, \eta)$	$f(\boldsymbol{\eta}, \boldsymbol{z}) = \dfrac{2e^{(z_1+\cdots+z_d)}}{e^{2(z_1+\cdots+z_d)} + 1}$.

Table 7: Generating functions of special d-tuples of umbral monomials

The notions of similarity and uncorrelation are updated as follows. Two d-tuples $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ of umbral monomials are said to be similar if they represent the same sequence of multivariate moments. They are said to be uncorrelated if $E[\boldsymbol{\mu}^{\boldsymbol{i}_1}\boldsymbol{\nu}^{\boldsymbol{i}_2}] = E[\boldsymbol{\mu}^{\boldsymbol{i}_1}]E[\boldsymbol{\nu}^{\boldsymbol{i}_2}]$.

Multivariate Lévy processes are represented by d-tuples of umbral monomials.

Definition 4.1. *A stochastic process* $\{\boldsymbol{X}_t\}_{t\geq 0}$ *on* $\mathbb{R}^d$ *is a* multivariate Lévy process *if*

(i) $\boldsymbol{X}_0 = \boldsymbol{0}$ *a.s.*

(ii) *For all* $n \geq 1$ *and for all* $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n < \infty$, *the r.v.'s* $\boldsymbol{X}_{t_2} - \boldsymbol{X}_{t_1}, \boldsymbol{X}_{t_3} - \boldsymbol{X}_{t_2}, \ldots$ *are independent.*

(iii) *For all* $s \leq t$, $\boldsymbol{X}_{t+s} - \boldsymbol{X}_s \stackrel{d}{=} \boldsymbol{X}_t$.

(iv) *For all* $\varepsilon > 0$, $\lim_{h\to 0} P(|\boldsymbol{X}_{t+h} - \boldsymbol{X}_t| > \varepsilon) = 0$.

(v) $t \mapsto \boldsymbol{X}_t(\omega)$ *are right-continuous with left limits, for all* $\omega \in \Omega$, *with* Ω *the underlying sample space.*

As in the univariate case, the moment generating function of a multivariate Lévy process is $\varphi_{\boldsymbol{X}_1}(\boldsymbol{z}) = E\left[e^{\boldsymbol{z}\boldsymbol{X}_1^T}\right]$, with $\boldsymbol{z} \in \mathbb{R}^d$. Paralleling the univariate case, the generating function of a d-tuple $\boldsymbol{\mu}$ is

$$f(\boldsymbol{\mu}, \boldsymbol{z}) = 1 + \sum_{k\geq 1} \sum_{\substack{\boldsymbol{i}\in\mathbb{N}_0^d \\ |\boldsymbol{i}|=k}} g_{\boldsymbol{i}} \frac{\boldsymbol{z}^{\boldsymbol{i}}}{\boldsymbol{i}!}.$$

Choose the d-tuple $\boldsymbol{\mu}$ such that $f(\boldsymbol{\mu}, \boldsymbol{z}) = \varphi_{\boldsymbol{X}_1}(\boldsymbol{z})$, that is $E[\boldsymbol{\mu}^{\boldsymbol{i}}] = E[\boldsymbol{X}_1^{\boldsymbol{i}}]$ for all $\boldsymbol{i} \in \mathbb{N}_0^d$. The auxiliary umbra $n.\boldsymbol{\mu}$ denotes the sum of n uncorrelated d-tuples of umbral monomials similar to $\boldsymbol{\mu}$. Its multivariate moment is [5]

$$E[(n.\boldsymbol{\mu})^{\boldsymbol{i}}] = \sum_{\boldsymbol{\lambda}\vdash\boldsymbol{i}} \frac{\boldsymbol{i}!}{\mathfrak{m}(\boldsymbol{\lambda})\boldsymbol{\lambda}!} (n)_{l(\boldsymbol{\lambda})} E[\boldsymbol{\mu}_{\boldsymbol{\lambda}}], \tag{4.2}$$

where $E[\boldsymbol{\mu}_{\boldsymbol{\lambda}}] = g_{\boldsymbol{\lambda}_1}^{r_1} g_{\boldsymbol{\lambda}_2}^{r_2} \ldots$ and $\boldsymbol{\lambda}$ is a partition[8] of the multi-index $\boldsymbol{i}$ of length $l(\boldsymbol{\lambda})$. By replacing the nonnegative integer n with the real parameter t in (4.2) the resulting auxiliary umbra $t.\boldsymbol{\mu}$ is the symbolic representation of the multivariate Lévy process $\boldsymbol{X}_t$.

As in the univariate case, since $\boldsymbol{\mu} \equiv \beta.\kappa_{\boldsymbol{\mu}}$ with $\kappa_{\boldsymbol{\mu}}$ the $\boldsymbol{\mu}$-cumulant umbra [13], a different representation for a multivariate Lévy process is $t.\beta.\kappa_{\boldsymbol{\mu}}$. The cumulant d-tuple could be further specified by using the multivariate Lévy-Khintchine formula [22].

Theorem 4.2. *$\boldsymbol{X} = \{\boldsymbol{X}_t\}_{t\geq 0}$ is a Lévy process if and only if there exists $\boldsymbol{m}_1 \in \mathbb{R}^d$, a symmetric, positive defined $d \times d$ matrix $\Sigma > 0$ and a measure ν on $\mathbb{R}^d$ with*

$$\nu(\{0\}) = 0 \text{ and } \int_{\mathbb{R}} (|\boldsymbol{x}|^2 \wedge 1)\nu(d\boldsymbol{x}) < \infty$$

such that

$$\varphi_{\boldsymbol{X}}(\boldsymbol{z}) = \exp\left\{ t \left[\frac{1}{2}\boldsymbol{z}\Sigma\boldsymbol{z}^T + \boldsymbol{m}_1\boldsymbol{z}^T + \int_{\mathbb{R}^d} (e^{\boldsymbol{x}\boldsymbol{z}^T} - 1 - \boldsymbol{x}\boldsymbol{z}^T \mathbf{1}_{\{|\boldsymbol{x}|\leq 1\}}(\boldsymbol{x}))\, \nu(\mathrm{d}\boldsymbol{x}) \right] \right\}. \tag{4.3}$$

The representation of $\varphi_{\boldsymbol{X}}(\boldsymbol{z})$ in (4.3) by $\boldsymbol{m}_1$, Σ and ν is unique.

Set $\boldsymbol{m}_2\boldsymbol{z}^T = \int_{\mathbb{R}^d} \boldsymbol{z}\boldsymbol{x}^T \mathbf{1}_{\{|\boldsymbol{x}|>1\}}(\boldsymbol{x})\, \nu(d\boldsymbol{x})$ and $\boldsymbol{m} = \boldsymbol{m}_1 + \boldsymbol{m}_2$, then

$$\varphi_{\boldsymbol{X}}(\boldsymbol{z}) = \exp\left\{ t \left[\frac{1}{2}\boldsymbol{z}\Sigma\boldsymbol{z}^T + \boldsymbol{m}\boldsymbol{z}^T + \int_{\mathbb{R}^d} (e^{\boldsymbol{z}\boldsymbol{x}^T} - 1 - \boldsymbol{z}\boldsymbol{x}^T)\, \nu(d\boldsymbol{x}) \right] \right\},$$

that is,

$$\varphi_{\boldsymbol{X}}(\boldsymbol{z}) = \exp\left\{ t \left[\frac{1}{2}\boldsymbol{z}\Sigma\boldsymbol{z}^T + \boldsymbol{m}\boldsymbol{z}^T \right] \right\} \exp\left\{ t \left[\int_{\mathbb{R}^d} (e^{\boldsymbol{z}\boldsymbol{x}^T} - 1 - \boldsymbol{z}\boldsymbol{x}^T)\, \nu(d\boldsymbol{x}) \right] \right\}. \tag{4.4}$$

Theorem 4.3. *Every Lévy process $\{\boldsymbol{X}_t\}_{t\geq 0}$ on $\mathbb{R}^d$ is umbrally represented by the family of auxiliary umbrae*

$$\{t.\beta.(\chi.\boldsymbol{m} \dot{+} \boldsymbol{\varsigma} C^T \dot{+} \boldsymbol{\eta})\}_{t\geq 0}, \tag{4.5}$$

where β is the Bell umbra, $\boldsymbol{m} \in \mathbb{R}^d$, $\boldsymbol{\varsigma}$ is the multivariate umbral counterpart of a standard gaussian r.v., C is the square root of the covariance matrix Σ and $\boldsymbol{\eta}$ is the multivariate umbra associated to the Lévy measure.

Every auxiliary umbra $t.\beta.\kappa_{\boldsymbol{\mu}}$ is the symbolic version of a multivariate compound Poisson r.v. of parameter t, that is a random sum $S_N = \boldsymbol{Y}_1 + \cdots + \boldsymbol{Y}_N$ of independent and identically distributed random vectors $\{\boldsymbol{Y}_i\}$, whose index N

[8] A partition $\boldsymbol{\lambda}$ of a multi-index $\boldsymbol{i}$, in symbols $\boldsymbol{\lambda} \vdash \boldsymbol{i}$, is a matrix $\boldsymbol{\lambda} = (\lambda_{ij})$ of nonnegative integers and with no zero columns in lexicographic order $\prec$ such that $\lambda_{r_1} + \lambda_{r_2} + \cdots + \lambda_{r_k} = i_r$ for $r = 1, 2, \ldots, d$. The number of columns of $\boldsymbol{\lambda}$ is denoted by $l(\boldsymbol{\lambda})$. The notation $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_1^{r_1}, \boldsymbol{\lambda}_2^{r_2}, \ldots)$ represents the matrix $\boldsymbol{\lambda}$ with r_1 columns equal to $\boldsymbol{\lambda}_1$, r_2 columns equal to $\boldsymbol{\lambda}_2$ and so on, where $\boldsymbol{\lambda}_1 \prec \boldsymbol{\lambda}_2 \prec \ldots$. We set $\mathfrak{m}(\boldsymbol{\lambda}) = (r_1, r_2, \ldots)$, $\mathfrak{m}(\boldsymbol{\lambda})! = r_1! r_2! \cdots$ and $\boldsymbol{\lambda}! = \boldsymbol{\lambda}_1! \boldsymbol{\lambda}_2! \cdots$.

is a Poisson r.v. of parameter t. Then the same holds for the Lévy process. The d-tuple $(\boldsymbol{\varsigma} C^T \dot{+} \chi.\boldsymbol{m} \dot{+} \boldsymbol{\eta})$ umbrally represents any of the random vectors $\{\boldsymbol{Y}_i\}$. Observe that $\chi.\boldsymbol{m}$ has not a probabilistic counterpart. If $\boldsymbol{m}$ is not equal to the zero vector, this parallels the well-known difficulty to interpret the Lévy measure as a probability measure.

4.1 Multivariate TSH polynomials.

The conditional evaluation with respect to an umbral d-tuple $\boldsymbol{\mu}$ has been introduced in [9]. Assume $\mathcal{X} = \{\mu_1, \mu_2, \ldots, \mu_d\}$. The conditional evaluation with respect to the umbral d-tuple $\boldsymbol{\mu}$ is the linear operator

$$E(\,\cdot \,|\, \boldsymbol{\mu}) : \mathbb{R}[x_1, \ldots, x_d][\mathcal{A}] \longrightarrow \mathbb{R}[\mathcal{X}]$$

such that $E(1 \,|\, \boldsymbol{\mu}) = 1$ and

$$E(x_1^{l_1}\, x_2^{l_2} \, \cdots \, x_d^{l_d}\, \boldsymbol{\mu}^{\boldsymbol{i}}\, \boldsymbol{\nu}^{\boldsymbol{j}}\, \boldsymbol{\eta}^{\boldsymbol{k}} \cdots \,|\, \boldsymbol{\mu}) = x_1^{l_1}\, x_2^{l_2} \, \cdots \, x_d^{l_d}\, \boldsymbol{\mu}^{\boldsymbol{i}}\, E[\boldsymbol{\nu}^{\boldsymbol{j}}]\, E[\boldsymbol{\eta}^{\boldsymbol{k}}] \cdots \tag{4.6}$$

for uncorrelated d-tuples $\boldsymbol{\mu}, \boldsymbol{\nu}, \boldsymbol{\eta} \ldots$, multi-indices $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k} \ldots \in \mathbb{N}_0^d$ and $\{l_i\}_{i=1}^d$ nonnegative integers. Since $f[(n+m).\boldsymbol{\mu}, \boldsymbol{z}] = f(\boldsymbol{\mu}, \boldsymbol{z})^{n+m} = f(n.\boldsymbol{\mu}, \boldsymbol{z})\, f(m.\boldsymbol{\mu}, \boldsymbol{z})$, then $(n+m).\boldsymbol{\mu} \equiv n.\boldsymbol{\mu} + m.\boldsymbol{\mu}'$, with $\boldsymbol{\mu}$ and $\boldsymbol{\mu}'$ uncorrelated d-tuples of umbral monomials. Then, for $E(\,\cdot \,|\, n.\boldsymbol{\mu})$ we assume $E[\{(n+m).\boldsymbol{\mu}\}^{\boldsymbol{i}} \,|\, n.\boldsymbol{\mu}] = E[\{n.\boldsymbol{\mu} + m.\boldsymbol{\mu}'\}^{\boldsymbol{i}} \,|\, n.\boldsymbol{\mu}]$ for all nonnegative integers n, m and for all $\boldsymbol{i} \in \mathbb{N}_0^d$. If $n \neq m$, then

$$E[\{(n+m).\boldsymbol{\mu}\}^{\boldsymbol{i}} \,|\, n.\boldsymbol{\mu}] = E[\{n.\boldsymbol{\mu} + m.\boldsymbol{\mu}\}^{\boldsymbol{i}} \,|\, n.\boldsymbol{\mu}], \tag{4.7}$$

since $n.\boldsymbol{\mu}$ and $m.\boldsymbol{\mu}$ are uncorrelated auxiliary umbrae. We will use the same d-tuple $\boldsymbol{\mu}$ as in (4.7) when no misunderstanding occurs. Thanks to equations (4.6) and (4.7), we have

$$E\left[\{(n+m).\boldsymbol{\mu}\}^{\boldsymbol{i}} \,\middle|\, n.\boldsymbol{\mu}\right] = \sum_{\boldsymbol{k} \leq \boldsymbol{i}} \binom{\boldsymbol{i}}{\boldsymbol{k}} (n.\boldsymbol{\mu})^{\boldsymbol{k}} E[(m.\boldsymbol{\mu})^{\boldsymbol{i}-\boldsymbol{k}}], \tag{4.8}$$

where $\boldsymbol{k} \leq \boldsymbol{i} \;\Leftrightarrow\; k_j \leq i_j$ for all $j = 1, \ldots, d$ and $\binom{\boldsymbol{k}}{\boldsymbol{i}} = \binom{k_1}{i_1} \cdots \binom{k_d}{i_d}$. By analogy with (4.7) and (4.8), we have $t.\boldsymbol{\mu} \equiv s.\boldsymbol{\mu} + (t-s).\boldsymbol{\mu}$ and for $t \geq 0$ and $s \leq t$

$$E\left[(t.\boldsymbol{\mu})^{\boldsymbol{i}} \,\middle|\, s.\boldsymbol{\mu}\right] = \sum_{\boldsymbol{k} \leq \boldsymbol{i}} \binom{\boldsymbol{i}}{\boldsymbol{k}} (s.\boldsymbol{\mu})^{\boldsymbol{k}} E[\{(t-s).\boldsymbol{\mu}\}^{\boldsymbol{i}-\boldsymbol{k}}].$$

Theorem 4.4. *For all $\boldsymbol{i} \in \mathbb{N}_0^d$, the family of polynomials*

$$Q_{\boldsymbol{i}}(\boldsymbol{x}, t) = E[(\boldsymbol{x} - t.\boldsymbol{\mu})^{\boldsymbol{i}}] \in \mathbb{R}[x_1, \ldots, x_d] \tag{4.9}$$

is TSH with respect to $\{t.\boldsymbol{\mu}\}_{t \geq 0}$.

The auxiliary umbra $-t.\boldsymbol{\mu}$ denotes the inverse of $t.\boldsymbol{\mu}$ that is $-t.\boldsymbol{\mu} + t.\boldsymbol{\mu} \equiv \boldsymbol{\epsilon}$, where $\boldsymbol{\epsilon}$ is the d-tuple such that $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \ldots, \epsilon_d)$ with $\{\epsilon_i\}$ uncorrelated augmentation umbrae. Coefficients of $Q_{\boldsymbol{i}}(\boldsymbol{x}, t)$ in (4.9) are such that

$$Q_{\boldsymbol{i}}(\boldsymbol{x}, t) = \sum_{\boldsymbol{k} \leq \boldsymbol{i}} \binom{\boldsymbol{i}}{\boldsymbol{k}} \boldsymbol{x}^{\boldsymbol{i}-\boldsymbol{k}} E[(-t.\boldsymbol{\mu})^{\boldsymbol{k}}]$$

so when $\boldsymbol{x}$ is replaced by $t.\boldsymbol{\mu}$ their overall evaluation is zero. Properties on the coefficients of $Q_{\boldsymbol{i}}(\boldsymbol{x}, t)$ can be found in [9]. Here we just recall a characterization of the coefficients of any multivariate TSH polynomial in terms of those of $Q_{\boldsymbol{i}}(\boldsymbol{x}, t)$.

Theorem 4.5. *A polynomial*

$$P(\boldsymbol{x}, t) = \sum_{\boldsymbol{k} \leq \boldsymbol{v}} p_{\boldsymbol{k}}(t)\, \boldsymbol{x}^{\boldsymbol{k}} \tag{4.10}$$

is a TSH polynomial with respect to $\{t.\boldsymbol{\mu}\}_{t \geq 0}$ *if and only if*

$$p_{\boldsymbol{k}}(t) = \sum_{\boldsymbol{k} \leq \boldsymbol{i} \leq \boldsymbol{v}} \binom{\boldsymbol{i}}{\boldsymbol{k}} p_{\boldsymbol{k}}(0)\, E[(-t.\boldsymbol{\mu})^{\boldsymbol{i}-\boldsymbol{k}}], \qquad \textit{for } \boldsymbol{k} \leq \boldsymbol{v}. \tag{4.11}$$

Table 8 and 9 give some examples of multivariate TSH polynomials and their connection with multivariate Lévy processes. The corresponding d-tuples are given in Table 7.

multivariate Lévy process	Umbral representation	TSH polynomial $Q_{\boldsymbol{i}}(\boldsymbol{x}, t)$
Brownian motion with covariance $\Sigma = CC^T$	$t.\beta.(\boldsymbol{\varsigma} C^T)$	$E[(\boldsymbol{x} - t.\beta.(\boldsymbol{\varsigma} C^T))^{\boldsymbol{i}}]$
$\boldsymbol{X}_t$ with $\boldsymbol{X}_1 \stackrel{d}{=} (U, \ldots, U)$ and U uniform r.v. on $[0,1]$	$-t.\boldsymbol{\iota}$	$E[(\boldsymbol{x} + t.\boldsymbol{\iota})^{\boldsymbol{i}}]$
$\boldsymbol{X}_t$ with $\boldsymbol{X}_1 \stackrel{d}{=} (Y, \ldots, Y)$ and Y Bernoulli r.v. of parameter $1/2$	$\frac{1}{2}[t.(\boldsymbol{u} - 1.\boldsymbol{\eta})]$	$E\left\{\left(\boldsymbol{x} + \frac{1}{2}[t.(\boldsymbol{\eta} - \boldsymbol{u})]\right)^{\boldsymbol{i}}\right\}$

Table 8: TSH polynomials associated to special multivariate Lévy processes

Let us remark that Hermite polynomials $H_{\boldsymbol{i}}^{(t^2)}(\boldsymbol{x}, \Sigma)$ in Table 9 are a generalization of the polynomials $H_{\boldsymbol{i}}(\boldsymbol{x})$ in [29] whose moment representation is $H_{\boldsymbol{i}}(\boldsymbol{x}) = E[(\boldsymbol{x}\Sigma^{-1} + \mathrm{i}\boldsymbol{Y})^{\boldsymbol{i}}]$ with E the expectation symbol, $\boldsymbol{Y} \simeq N(\boldsymbol{0}, \Sigma^{-1})$ and Σ a covariance matrix of full rank d.

A generalization of Lévy-Sheffer system to the multivariate case has been introduced in [9]. A sequence of multivariate polynomials $\{V_{\boldsymbol{k}}(\boldsymbol{x}, t)\}_{t \geq 0}$ is a

multivariate Lévy process	Special family of polynomials	Connection with TSH polynomials
Brownian motion with covariance $\Sigma = CC^T$	Hermite $H_{\boldsymbol{i}}^{(t^2)}(\boldsymbol{x}, \Sigma)$	$H_{\boldsymbol{i}}^{(t^2)}(\boldsymbol{x}, \Sigma) = Q_{\boldsymbol{i}}(\boldsymbol{x}, t)$
$\boldsymbol{X}_t$ with $\boldsymbol{X}_1 \stackrel{d}{=} (U, \ldots, U)$ and U uniform r.v. on $[0, 1]$	Bernoulli $B_{\boldsymbol{i}}^{(t)}(\boldsymbol{x})$	$B_{\boldsymbol{i}}^{(t)}(\boldsymbol{x}) = Q_{\boldsymbol{i}}(\boldsymbol{x}, t)$
$\boldsymbol{X}_t$ with $\boldsymbol{X}_1 \stackrel{d}{=} (Y, \ldots, Y)$ and Y Bernoulli r.v. of parameter $1/2$	Euler $\mathcal{E}_{\boldsymbol{i}}^{(t)}(\boldsymbol{x})$	$\mathcal{E}_{\boldsymbol{i}}^{(t)}(\boldsymbol{x}) = Q_{\boldsymbol{i}}(\boldsymbol{x}, t)$

Table 9: Special families of polynomials and TSH polynomials

multivariate Lévy-Sheffer system if

$$1 + \sum_{k \geq 1} \sum_{\substack{\boldsymbol{v} \in \mathbb{N}_0^d \\ |\boldsymbol{v}| = k}} V_{\boldsymbol{k}}(\boldsymbol{x}, t) \frac{\boldsymbol{z}^{\boldsymbol{k}}}{\boldsymbol{k}!} = [g(\boldsymbol{z})]^t \exp\{(x_1 + \cdots + x_d)[h(\boldsymbol{z}) - 1]\},$$

where $g(\boldsymbol{z})$ and $h(\boldsymbol{z})$ are analytic in a neighborhood of $\boldsymbol{z} = \boldsymbol{0}$ and

$$\left.\frac{\partial}{\partial z_i} h(\boldsymbol{z})\right|_{\boldsymbol{z}=\boldsymbol{0}} \neq 0 \quad \text{for } i = 1, 2, \ldots, d.$$

If $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ are d-tuples of umbral monomials such that $f(\boldsymbol{\mu}, \boldsymbol{z}) = g(\boldsymbol{z})$ and $f(\boldsymbol{\nu}, \boldsymbol{z}) = h(\boldsymbol{z})$ respectively, then

$$V_{\boldsymbol{k}}(\boldsymbol{x}, t) = E[(t.\boldsymbol{\mu} + (x_1 + \cdots + x_d).\beta.\boldsymbol{\nu})^{\boldsymbol{k}}]. \tag{4.12}$$

The multivariate Lévy-Sheffer polynomials for the pair $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ are TSH polynomials with respect to a special symbolic multivariate Lévy process involving the multivariate compositional inverse of a d-tuple $\boldsymbol{\nu}$. Assume $\boldsymbol{\chi}_{(i)}$ the d-tuple with all components equal to the augmentation umbra and only the i-th one equal to the singleton umbra, that is $\boldsymbol{\chi}_{(i)} = (\epsilon, \ldots, \chi, \ldots, \epsilon)$. The multivariate compositional inverse of $\boldsymbol{\nu}$ is the umbral d-tuple $\boldsymbol{\nu}^{<-1>} = ((\boldsymbol{\nu}^{<-1>})_1, \ldots, (\boldsymbol{\nu}^{<-1>})_d)$ such that $(\boldsymbol{\nu}^{<-1>})_i.\beta.\boldsymbol{\nu} \equiv \boldsymbol{\chi}_{(i)}$ for $i = 1, \ldots, d$.

Theorem 4.6. *The multivariate Lévy-Sheffer polynomials for the pair $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ are TSH polynomials with respect to the symbolic multivariate Lévy process*

$$\{t.(\mu_1.\beta.\boldsymbol{\nu}_1^{<-1>} + \cdots + \mu_d.\beta.\boldsymbol{\nu}_d^{<-1>})\}_{t \geq 0}.$$

5 Conclusions and open problems.

In this paper, the review of a symbolic treatment of TSH polynomials, relied on the classical umbral calculus, is proposed. The main advantage of this symbolic presentation is the plainness of the overall setting which reduces to few

fundamental statements, but also the availability of efficient routines [7] for the implementation of formulae as (4.2), which is the key to manage the polynomials $Q_{\boldsymbol{k}}(\boldsymbol{x}, t)$.

The main result of this presentation is that any univariate (respectively multivariate) TSH polynomial has the form $Q_k(x,t)$ (respectively $Q_{\boldsymbol{k}}(\boldsymbol{x}, t)$) or can be expressed as a linear combination of the polynomials $Q_k(x,t)$ with coefficients given by (4.11). Thanks to the umbral representation of multivariate Lévy-Sheffer systems, more families of umbral polynomials could be characterized, together with their orthogonality properties. This will be the object of future research and investigation.

In [2], Barrieu and Shoutens have related the infinitesimal generator of a Markov process to a more general class of linear operators possessing the TSH property, both ascribable to special families of martingales. A stochastic Taylor formula is produced which results to be a generalization of a TSH polynomial due to the presence of a remainder term series. A symbolic representation of this new TSH function could open the way to a new classification of the corresponding operators by which to recover the martingale property on Lévy processes. Similarly, the extension to the more general class of Markov processes (a first attempt is given in [2]) would move the employment of TSH functions beyond the field of applications strictly connected to the market portfolio. One step more consists in dealing with matrix-valued stochastic processes by replacing formal power series (2.9) with hypergeometric functions, as done in [18]. This would allow us a symbolic representation also for zonal polynomials whose computational handling is still an open problem.

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